

GROUP ACTIONS ON S^6 AND COMPLEX STRUCTURES ON \mathbb{P}_3

ALAN T. HUCKLEBERRY, STEFAN KEBEKUS AND THOMAS PETERNELL

ABSTRACT. It is proved that if S^6 possesses an integrable complex structure, then there exists a 1-dimensional family of pairwise different exotic complex structures on $\mathbb{P}_3(\mathbb{C})$. This follows immediately from the main result of the paper: S^6 is not the underlying differentiable manifold of an almost homogeneous complex manifold X . Via elementary Lie theoretic techniques this is reduced to ruling out the possibility of a \mathbb{C}^* -action on a certain non-normal surface $E \subset X$. A contradiction is reached by analyzing combinatorial aspects of the non-normal locus N of E and its preimage \tilde{N} in the normalization \hat{E} .

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1. INTRODUCTION

This note is motivated by the following classical problem: Is there a complex structure on the 6-sphere S^6 , i.e. is S^6 a complex manifold?

It has been known since decades that all other spheres S^{2n} do not even admit an almost complex structure. On the other hand S^6 admits many almost complex structures. It is generally believed that none of them is integrable.

Suppose S^6 has a complex structure X . Then by [CDP98] every meromorphic function on X is constant. Moreover X is not Kähler, since $b_2(X) = 0$. Therefore the problem is quite inaccessible by standard methods of complex geometry.

In this paper we prove

Theorem 1.1. *X is not almost homogeneous. In other words, the automorphism group $\text{Aut}_{\mathcal{O}}(X)$ does not have an open orbit.*

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This is related as follows to the question of existence of complex structures on the underlying differentiable manifold of $\mathbb{P}_3(\mathbb{C})$: As above assume $X = S^6$ has the structure of a complex manifold. For $p \in X$ let $\pi_p : X_p \rightarrow X$ denote the blow up of X at p with $\pi_p^{-1}(p) =: E_p$. Of course $E_p = \mathbb{P}_2(\mathbb{C})$. Since sufficiently small neighborhoods of a hyperplane in $\mathbb{P}_n(\mathbb{C})$ are differentiably identifiable with neighborhoods of a blown up point, it follows that X_p is diffeomorphic to $\mathbb{P}_3(\mathbb{C})$.

Suppose that for given points $p, q \in X$ there exists a biholomorphic mapping $\psi : X_p \rightarrow X_q$. Note that, since $\psi(E_p)$ generates the cohomology $H^2(X_q, \mathbb{Z})$, it follows that $\psi(E_p) \cap E_q \neq \emptyset$. If $C := \psi(E_p) \cap E_q \neq E_q$, then $\pi_q|_{\psi(E_p)}$ would be a modification which maps the curve $C \subset \psi(E_p) \cong \mathbb{P}_2(\mathbb{C})$ to a point. Since this is impossible, $\psi(E_p) = E_q$ and ψ induces an automorphism $g_\psi : X \rightarrow X$ with $g_\psi(p) = q$.

Let \mathcal{F} denote the orbit space $X/\text{Aut}_{\mathcal{O}}(X)$. For $\xi, \eta \in \mathcal{F}$ with $\xi \neq \eta$ and $p \in \xi, q \in \eta$ representatives, it follows that X_p and X_q are not biholomorphically equivalent.

Of course \mathcal{F} may very well be non-Hausdorff and, from certain points of view, an unreasonable parameter space, but by abuse of language we nevertheless refer to it as a family of complex structures on $\mathbb{P}_3(\mathbb{C})$. Since $\text{Aut}_{\mathcal{O}}(X)$ does not have an open orbit, its generic orbit in X is at least 1-codimensional. In particular, for general p , the semi-universal deformation space of X_p is at least 1-dimensional. In this sense we have the following consequence of theorem 1.1.

Corollary 1.2. *If S^6 admits a complex structure, then there is a 1-dimensional family of complex structures on \mathbb{P}_3 .*

Corollary 1.3. *Let X be a complex structure on S^6 . Then X carries at most two linearly independent holomorphic vector fields: $h^0(TX) \leq 2$.*

Proof. If the generic $\text{Aut}_{\mathcal{O}}(X)$ -orbit is 2-dimensional, then there are globally defined holomorphic vector fields V_1 and V_2 such that $U := \{p \in X : V_1 \wedge V_2(p) \neq 0\}$ is a dense, Zariski open subset. If V_3 is any holomorphic field on X , then, since $V_1 \wedge V_2 \wedge V_3 \equiv 0$, there exist $m_1, m_2 \in \mathcal{O}(U)$ with $V_3 = m_1 V_1 + m_2 V_2$ on U . By explicit computation in local coordinates, one verifies that these coefficients are in fact meromorphic on X and the relation between the fields extends to X . On the other hand, it has been proven in [CDP98] that X possesses only the constant meromorphic functions. Thus, it follows in this case that $\dim_{\mathbb{C}} \Gamma(X, TX) = 2$. The other cases, i.e., those where the generic orbit dimension is smaller, are handled analogously. \square

By semicontinuity we obtain from Corollary 1.3:

Corollary 1.4. *The inequality $h^0(TX \otimes L) \leq 2$ holds for generic $L \in \text{Pic}^0(X)$.*

Our theorem should be seen in a more general context or rather program for investigating complex structures on S^6 . Namely we would like to prove that the tangent bundle TX is stable with respect to a Gauduchon metric. It would then follow that X carries a Hermite-Einstein connection (Li-Yau). At this point one could employ powerful analytic tools to investigate the problem further. In order to prove the stability it would seem necessary to verify the statements:

- (A): $H^0(X, TX \otimes L) = 0$ for all $L \in \text{Pic}^0(X)$;
- (B): $H^0(X, \Omega_X^1 \otimes L) = 0$ for all $L \in \text{Pic}^0(X)$.

Hence our theorem is the first approach to (A). The next step should be to investigate group actions in general. We feel that the study of group actions on highly non-algebraic manifolds is of independent interest and hope that the methods which we develop in this paper are useful in a broader context.

2. SETUP AND GENERAL RESULTS

In this section we gather results which will be used throughout the paper and give the general setup.

2.1. Setup and outline of proof. Let X denote a complex structure on S^6 . The entire paper is devoted to proving that X is not almost homogeneous. This is proved by assuming the contrary, i.e., there exists $x_0 \in X$ such that $G.x_0 =: \Omega$ is open, and deriving a contradiction.

The structure of the proof can be outlined as follows: Using the fact that X has only finitely many analytic hypersurfaces in connection with topology and Lie theory of the situation at hand, it is shown in sections 3 that if G exists, then it must be solvable. In section 4, we give a number of methods which are applied in section 5 in order to rule out this case, too.

Two of the methods involve a lengthy proof which we have preferred to give separately in section 6 and sections 7–9.

The technical heart of this work lies in sections 7–9 where we rule out the following situation: we suppose that there exists of a subgroup $\mathbb{C}^* < \text{Aut}_{\mathcal{O}}(X)$ and that $E := X \setminus \Omega$ is an irreducible, non-normal, rational surface where \mathbb{C}^* acts as an algebraic transformation group.

The non-normal locus $N \subset E$ and its preimage \tilde{N} in the normalization \tilde{E} play a key role in the remainder of the proof. It follows from the Betti number information on X that N and \tilde{N} are connected and have the same first Betti numbers. An analysis of the \mathbb{C}^* -action on E shows however that in fact $b_1(N) = b_1(\tilde{N}) + 1$ (see sections 8–9).

2.2. Meromorphic functions, discrete isotropy, and the dimension of E . Since there are no non-constant meromorphic function on X by [CDP98], we have that $\dim G = h^0(TX) = 3$.

Proposition 2.1. *The G -isotropy Γ at a point $x_0 \in \Omega$ is discrete. Furthermore, $E := X \setminus \Omega$ is non-empty and 1-codimensional in X .*

Proof. Since $\dim G = 3$, it is clear that Γ is discrete. In particular, if G would act transitively, then, contrary to assumption, $\pi_1(X) \cong \Gamma$ would not be trivial. Thus $E \neq \emptyset$. If the vector fields X_1, X_2, X_3 form a basis of $\Gamma(X, TX)$, then

$$E = \{X_1 \wedge X_2 \wedge X_3 = 0\}.$$

In particular, $-K_X = \mathcal{O}_X(\sum \lambda_i E_i)$, where E_i are the irreducible components of E and $\lambda_i > 0$. \square

Let $G = R.S$ be the Levi-Malcev decomposition, so that R is the radical of G , i.e. the maximal connected solvable normal subgroup of G and S is semisimple; moreover $R \cap S$ is discrete.

Since a semi-simple complex Lie group has dimension at least 3, we have only two cases, namely that G is semi-simple or solvable.

2.3. Topology of Ω and E . Since the topology of X is well-known, the Betti-numbers of Ω and E are closely related.

Notation 2.2. If Y is a complex space, set $b_i(Y) := h^i(Y; \mathbb{Q})$.

Proposition 2.3. *For $q \in \{1, \dots, 4\}$, the following equation of Betti-numbers holds: $b_q(E) = b_{5-q}(\Omega)$.*

Proof. Recall from algebraic topology that there is an exact cohomology sequence associated to the pair (X, E) :

$$\dots \rightarrow H^q(X; \mathbb{Q}) \rightarrow H^q(E; \mathbb{Q}) \rightarrow H^{q+1}(X, E; \mathbb{Q}) \rightarrow H^{q+1}(X, \mathbb{Q}) \rightarrow \dots$$

Since X is homeomorphic to the 6-sphere, $b_q(X) = b_{q+1}(X) = 0$ for all numbers $1 \leq q \leq 4$, so that $H^q(E; \mathbb{Q}) \cong H^{q+1}(X, E; \mathbb{Q})$. An application of the Alexander duality theorem yields $H^{q+1}(X, E; \mathbb{Q}) \cong H_{5-q}(\Omega; \mathbb{Q})$, hence the claim. \square

2.4. Fixed points of reductive groups. Many of our arguments involve linearization of group actions at fixed points. We recall the theorem on faithful linearization:

Theorem 2.4. *Assume that a reductive complex Lie group H acts holomorphically on a complex manifold M , and assume that $x \in M$ is an H -fixed point. For $h \in H$, let $T(h) : T_x \rightarrow T_x$ be the tangential map. Then there exist neighborhoods U of x , and V of $0 \in T_x M$ and an isomorphism $\phi : U \rightarrow V$ such that $\phi \circ h = T(h) \circ \phi$ for all h in a given maximal compact subgroup of H .*

Furthermore, if W is a neighborhood of the maximal compact subgroup and $U' \subset U$ open so that $WU' \subset U$, then $(T(w) \circ \phi)(x) = (\phi \circ w)(x)$ for all $x \in U'$.

In this setting we call U a *linearizing neighborhood* of x . See [Huc90] or [HO80, p. 11f] information about linearization.

The fixed point set of a reductive group also possesses certain topological properties. In our special case this implies:

Proposition 2.5. *Suppose that $\text{Aut}(X)$ contains a subgroup $I \cong \mathbb{C}^*$. Let F be the fixed point set of I . Then either $F \cong \mathbb{P}_1$, or F consists of two disjoint points.*

Proof. As a first point, note that it follows from the linearization theorem that F is smooth; in particular, its irreducible components are disjoint. Furthermore, $\chi_{\text{Top}}(F) = \chi_{\text{Top}}(X)$ (see [KP85] for a proof in the algebraic setting which carries over immediately to, e.g., compact complex spaces.)

In our situation, for $p \in F$ the group I stabilizes the complement $X \setminus \{p\} \cong S^6$ and thus F is acyclic (see [Bre72]). Consequently, F consists of two points or it is irreducible. Thus it remains to show that F is at most 1-dimensional.

Suppose F is 2-dimensional. Since $H^4(X, \mathbb{Z}) = 0$, it follows that $c_2(TX|_F) = 0$ and, since $H^2(X, \mathbb{Z}) = 0$, the normal bundle $N_{F,X}$ is topologically trivial. Consequently

$$0 = c_2(TX|_F) = c_2(F) + N_{F,X} \cdot K_F = \chi_{\text{Top}}(F)$$

which is contrary to $\chi_{\text{Top}}(F) = 2$. \square

3. THE CASE WHERE G IS SEMISIMPLE

In this section we treat the case that G is semisimple. Since $\dim G = 3$, it follows that $G \cong SL_2$. The most basic property of almost transitive SL_2 -actions on threefolds is:

Lemma 3.1. *The G -action on X does not have a fixed point. In particular, if \tilde{E}_i is the normalization of E_i , then \tilde{E}_i is smooth.*

Proof. Assume that $x \in X$ was a G -fixed point. Linearize the G -action at x and recall from the representation theory of SL_2 that no 3-dimensional SL_2 -representation space is SL_2 -almost homogeneous (see [MU83, lemma 1.12] for a more detailed proof). A contradiction. \square

Lemma 3.2. *If $E_i \subset E$ is an irreducible component, then G acts almost transitively on E_i . In particular, the normalization \tilde{E}_i is either rational or a Hopf-surface.*

Proof. If G did not have an open orbit in E_i , then by lemma 3.1 all G -orbits would be 1-dimensional. Thus the normalization \tilde{E} would be the product $\mathbb{P}_1(\mathbb{C}) \times C$, where G operates transitively on the first factor and C is a smooth curve. In particular it follows that a maximal torus $T \cong \mathbb{C}^*$ would have two disjoint copies of C as a fixed point set in E_i . Note that, since the normalization map $\tilde{E}_i \rightarrow E_i$ is equivariant with respect to $G \cong SL_2$, the T -fixed point set in E_i is the disjoint union of two curves, contrary to proposition 2.5.

The “In particular...” clause of the statement results from the classification of the almost homogeneous surfaces —see [HO80, p. 92] or [Pot69]. \square

Now we exclude both possibilities:

Proposition 3.3. *The group G is not semisimple.*

Proof. It follows from the above lemma that the normalization \tilde{E}_i of a component of E is an almost homogeneous Hirzebruch surface, $\mathbb{P}_2(\mathbb{C})$ equipped with either the defining representation of $SO_3(\mathbb{C})$ or the representation of SL_2 with a fixed point or a homogeneous Hopf surface (see [HO80] or [Pot69]). Consequently, a maximal torus $T \cong \mathbb{C}^* < G$ has only isolated fixed points in X and, if E_i were rational it would already have 3 or 4 fixed points in E_i alone.

Thus we may assume that every such component is a homogeneous Hopf surface. But this is also not possible, because such a surface is a homogeneous space G/H , where H^0 is unipotent, i.e., T has no fixed points. \square

4. MAIN METHODS FOR THE ELIMINATION OF SOLVABLE GROUPS

We begin by presenting several methods which involve the normalizer of subgroups of isotropy groups. Recall that G is a connected, simply-connected complex Lie group acting almost transitively on X with an open orbit $\Omega = G.x_0$. By proposition 3.3 we may assume that G is solvable and by the remarks in section 2 that the isotropy $\Gamma := G_{x_0}$ is discrete.

4.1. The normalizer arguments. Throughout the paper if H is a subgroup of G , then $N(H)$ denotes its normalizer in G and $N(H)^0$ its identity component. If H is discrete, it follows that $N(H)^0 = Z(H)^0$, where $Z(H)$ denotes its centralizer.

4.1.1. *The 2-dimensional normalizer argument.* Note that if $H < \Gamma$ is not normal in G , then it acts non-trivially on Ω .

Proposition 4.1 (2-dimensional normalizer argument). *If $H < \Gamma$ is an arbitrary subgroup, then $\dim N(H) \neq 2$.*

Proof. Suppose not and note that the 2-dimensional orbit $F := Z(H)^0.x_0$ is a component of the set of H -fixed points in Ω . Since the full set X^H of H -fixed points is closed, F is Zariski open in its closure \overline{F} which is 1-dimensional in X .

Observe that $\{g\overline{F} \mid g \in G\}$ is an infinite set of hypersurfaces and consequently there exists a non-constant meromorphic function on X (see [Kra75, thm. 1]¹). However, such a function does not exist (see [CDP98]). \square

4.1.2. *The 1-dimensional normalizer argument.*

Proposition 4.2 (1-dimensional normalizer argument). *If $H < \Gamma$ is any subgroup and $\dim N(H) = 1$, then $N(H)^0 \cap \Gamma$ is a lattice of rank 2.*

Proof. The 1-dimensional group $Z := Z(H)^0$ acts transitively on the fixed point components $\Omega^H \subset \Omega$ which are of course Zariski open in their closures. Set $C := Z.x_0$ and note that if $C \neq \overline{C}$, then $\overline{C}.E > 0$, contrary to $h^2(X; \mathbb{Z}) = 0$. Thus the orbit C is a compact 1-dimensional torus. \square

4.2. **Arguments involving reductive subgroups of $\text{Aut}(X)$.** In many cases we are able to rule out the existence of certain subgroups of $\text{Aut}(X)$ under additional assumptions on the topology of Ω . For convenience, use the following

Notation 4.3. If Y and Z are topological spaces, say that Y has “Betti-type Z ” if all the Betti-numbers of Y and Z agree.

4.2.1. *The $\mathbb{C}^* \times \mathbb{C}^*$ -action argument.*

Proposition 4.4. *If E is connected and has at least two irreducible components, then there does not exist an (effective) action of $(\mathbb{C}^*)^2$ on X .*

The proof will be given in section 6.

4.2.2. *The torus-action argument.*

Proposition 4.5 (torus-action argument). *The automorphism group of X does not contain a compact torus. In particular, if $\Gamma' < \Gamma$ is an arbitrary subgroup which is normal in G and contained in a 1-dimensional Abelian subgroup A , then $\text{rank}(\Gamma') = 1$.*

Proof. Assume that $T < \text{Aut}(X)$ is a 1-dimensional complex torus. Since $\chi_{\text{Top}}(X) = 2$, the Lefschetz fixed point formula shows that every vector field must have zeros. In particular, T must have a fixed point in X . On the other hand, all representations of T are trivial, a contradiction the theorem on faithful linearization! \square

¹See [FF79] for a more general result.

4.2.3. The \mathbb{C}^* -action argument.

Proposition 4.6. *If the open orbit $\Omega \subset X$ has the same Betti numbers as the circle S^1 , then the automorphism group of X does not contain a subgroup which is isomorphic to \mathbb{C}^* .*

The proof of the preceding proposition turns out to be a rather involved matter. We give it in sections 7–9.

The following corollary is useful when it comes to the exclusion of certain subgroups Γ .

Corollary 4.7 (\mathbb{C}^* -action argument). *If the open orbit $\Omega \subset X$ has Betti-type S^1 , and if $\Gamma' < \Gamma$ is an arbitrary subgroup which is normal in G , then Γ' is not contained in any positive-dimensional Abelian subgroup of G .*

Proof. Assume to the contrary: let $A < G$ be an Abelian, positive-dimensional connected subgroup and consider the A -action on X , given by a morphism $\rho : A \rightarrow \text{Aut}(X)$ of complex Lie groups.

Since it is normal, Γ' acts trivially on Ω , and thus trivially on X . Consequence: the morphism ρ factors through the quotient A/Γ' , i.e. there exists an action of A/Γ' on X .

Let $A' < A$ be a 1-dimensional subgroup such that $A' \cap \Gamma' \neq \{0\}$. Then $A'/(A' \cap \Gamma')$ acts on X . By proposition 4.6, $A'/(A' \cap \Gamma')$ must be a compact torus, but we have seen in proposition 4.5 that no compact torus acts non-trivially on X . \square

4.3. Topological observations. Recall that G is solvable and simply-connected, e.g. it is a cell, and $\Omega = G/\Gamma$, where Γ is discrete. Thus the topology of Ω is completely determined by Γ . It follows directly from proposition 2.3 that $\Gamma \neq \{e\}$.

4.3.1. Restriction on the Betti-type of Ω .

Proposition 4.8 (Betti-type argument). *The open orbit $\Omega \subset X$ does not have the Betti-type of a $S^1 \times S^1$.*

Proof. Assume to the contrary. Then, by proposition 2.3, we have $b_4(E) = 2$ and $b_3(E) = 1$. Similar to the proof of proposition 2.3, the sequence of the pair (E, Ω) and Alexander duality give

$$0 \rightarrow \underbrace{H^0(X, E; \mathbb{Q})}_{=H_6(\Omega; \mathbb{Q})=0} \rightarrow \underbrace{H^0(X; \mathbb{Q})}_{=\mathbb{Q}} \rightarrow H^0(E; \mathbb{Q}) \rightarrow \underbrace{H^1(X, E; \mathbb{Q})}_{=H_5(\Omega; \mathbb{Q})=0} \rightarrow \dots$$

so that $b_0(E) = 1$. In particular, E is connected and has exactly two irreducible components E_1 and E_2 . Now $E_1|_{E_2}$ is a Cartier divisor which is effective, non-zero, but homologically equivalent to zero, so that no desingularization of E_2 is Kähler and vice versa.

By the Mayer-Vietoris sequence

$$\dots \rightarrow H^3(E) \rightarrow H^3(E_1) \oplus H^3(E_2) \rightarrow \underbrace{H^3(E_1 \cap E_2)}_{=0} \rightarrow \dots$$

we obtain $b_3(E_1) + b_3(E_2) \leq b_3(E) = 1$. Hence we may assume that $b_3(E_1) = 0$.

Let $\nu : \tilde{E}_1 \rightarrow E_1$ be the normalization. Let $N \subset E$ be the (set-theoretical) non-normal locus and $\tilde{N} := \nu^{-1}(N)$.

The sequence

$$\dots \rightarrow H^q(E_1) \rightarrow H^q(\tilde{E}_1) \oplus H^q(N) \rightarrow H^q(\tilde{N}) \rightarrow \dots \quad q \geq 1$$

yields $b_3(E_1) \geq b_3(\tilde{E}_1)$ so that $b_3(\tilde{E}_1) = 0$ (see [BK82, prop. 3.A.7 on p. 98] for an explanation of the sequence).

Now consider the desingularization $\pi : \hat{E}_1 \rightarrow \tilde{E}_1$. Since $\pi_*(\mathbb{Z})$ is a skyscraper sheaf with support only finitely many points, $E_2^{p,q} := H^p(\tilde{E}_1, R^q\pi_*(\mathbb{Z})) = 0$ for all $p > 0$. Thus, $E_2^{p,q} \cong E_\infty^{p,q}$ in the Leray spectral sequence, and since $R^3\pi_*(\mathbb{Z}) = 0$ and $H^3(\tilde{E}_1, \mathbb{Z}) = 0$, we obtain $b_3(\hat{E}_1) = 0$. Thus $b_1(\hat{E}) = 0$, and the classification of surfaces yields that \hat{E}_1 is Kähler². This is a contradiction! \square

5. ELIMINATION OF THE SOLVABLE GROUPS

Our goal here is to eliminate the possibility that X is almost homogeneous with respect to the action of a solvable group. The proof requires a bit of the special knowledge given by the classification the of the simply connected 3-dimensional solvable Lie groups.

5.1. Classification of the relevant Lie groups. We now recall the classification mentioned above (see e.g. [Jac62]). In this case G is biholomorphic to \mathbb{C}^3 as a complex manifold and the group structure is isomorphic to one of the following:

G_0 : this is the well-known Abelian group \mathbb{C}^3 .

G_1 : we could also denote this group by $G_2(0)$. The multiplication is given as

$$(a_1, b_1, c_1)(a_2, b_2, c_2) = (a_1 + a_2, e^{a_1}b_2 + b_1, c_1 + c_2)$$

$G_2(\tau)$: here τ is any complex number other than zero. The multiplication is

$$(a_1, b_1, c_1)(a_2, b_2, c_2) = (a_1 + a_2, e^{a_1}b_2 + b_1, e^{\tau a_1}c_2 + c_1)$$

G_3 : Multiplication:

$$(a_1, b_1, c_1)(a_2, b_2, c_2) = (a_1 + a_2, e^{a_1}b_2 + b_1 + a_1e^{a_1}c_2, e^{a_1}c_2 + c_1)$$

H_3 : this is the Heisenberg group, where the multiplication is

$$(a_1, b_1, c_1)(a_2, b_2, c_2) = (a_1 + a_2, b_1 + b_2 + a_1c_2, c_1 + c_2)$$

For a detailed study of the discrete subgroups of such groups see [ES86, Sect. 1].

5.2. The elimination. Here we eliminate the possibility of an action of a solvable group by utilizing a general strategy along with some knowledge of the Lie groups which occur.

Proposition 5.1. *If Γ is normal in G and $A < G$ is any closed connected Abelian subgroup, then $\Gamma \not\subset A$. In particular, the group G is not isomorphic to $G_0 \cong \mathbb{C}^3$.*

Proof. Assume to the contrary, i.e., that $\Gamma \subset A$. Since G/A is acyclic, G/Γ has the same homotopy type as A/Γ , i.e., the same homotopy type as a real torus.

Our argument depends on $\text{rank}(\Gamma)$.

$\text{rank}(\Gamma) = 1$: Then Ω has the Betti-type of the circle S^1 . Take a 1-dimensional subgroup $A' < G$, $A' \cong \mathbb{C}$ with $\Gamma \subset A'$ and observe that $A/\Gamma \cong \mathbb{C}^*$ acts non-trivially on X , contrary to the \mathbb{C}^* -action argument 4.6.

$\text{rank}(\Gamma) = 2, 3, 4$: Betti number considerations show that the assumptions of proposition 4.5 are satisfied. However, the assumption on $\text{rank}(\Gamma)$ together with the torus-action argument allows us to construct a $(\mathbb{C}^*)^2$ -action coming from the group A .

²Actually it is easy to see that $\kappa(\hat{E}_1) = -\infty$ so that we do not have to use the fact that $K3$ -surfaces are Kähler.

rank(Γ) = 5: Here $G = A$ is Abelian and Ω has two ends, i.e., proposition 4.5 may not be applied. However, the ineffectivity of the G -action on every component of E must contain a reductive group, i.e. a torus or \mathbb{C}^* , contrary to either the torus-action argument or the characterization of the fixed point set of a \mathbb{C}^* action.

This finishes the proof. \square

A weaker statement holds if Γ is not necessarily normal.

Lemma 5.2. *For all groups $A < G$ with $A \cong \mathbb{C}$ we have $\Gamma \not\subset A$.*

Proof. Assume to the contrary. Again the Betti-types of G/Γ and of A/Γ agree. We may assume that Γ is not normal.

Since A is contained in the normalizer of Γ , the 2-dimensional normalizer argument implies that $A = N_G(\Gamma)$. In this setting the 1-dimensional normalizer argument yields that $\Gamma \cong \mathbb{Z}^2$, contrary to the Betti-type argument. \square

We will need the following technical lemma on centralizers of elements in G .

Lemma 5.3. *If $G \cong G_1, G_2(\tau), G_3$ or H_3 and $g \in G$, then $\dim Z_G(g) \geq 1$ and one of the following holds:*

1. $\dim Z_G(g) \geq 2$, or
2. $Z_G(g)^0 \subset (0, \mathbb{C}, \mathbb{C})$, or
3. $Z_G(g)^0 = Z_G(g)$, i.e. $Z_G(g)$ is connected.

Here $Z_G(g)$ denotes the set of elements commuting with g .

Proof. The statement that $\dim Z_G(g) \geq 1$ can be checked directly. To show that one of (1), (2) or (3) holds, for any number $a \in \mathbb{C}$ define the set

$$Z_a := \{(b, c) \in \mathbb{C}^2 \mid (a, b, c) \in G \text{ commutes with } g\}.$$

Note that it is sufficient to show that for all a either $\dim Z_a > 0$ or $\#Z_a \in \{0, 1\}$ holds. The last statement follows by an elementary calculation of commutators. \square

Now we start with the

Proposition 5.4. *The group G is not solvable.*

Proof. By proposition 5.1, assume that $G \cong G_1, G_2(\tau), G_3$ or H_3 .

If $G \cong G_1$ or H_3 , then for all $g \in G$ we have that $\dim Z_G(g) \geq 2$. Thus, by the 2-dimensional normalizer argument, $\Gamma < Z_G$. A direct calculation shows that Z_G is contained in a connected Abelian subgroup, contrary to proposition 5.1.

Now assume that $G \cong G_2(\tau)$ or G_3 . Note that $G \cong A \ltimes G'$ where $G' = (0, \mathbb{C}, \mathbb{C})$ is the commutator group of G and A as well as G' are connected and Abelian. Furthermore, $\dim Z_G(g') = 2$ for all $g' \in G'$. Thus, by the 2-dimensional normalizer argument $\Gamma \cap G' = \{e\}$, and the projection $\pi_1 : G \rightarrow A$ is an injective group morphism, if restricted to Γ . This shows that Γ is Abelian, which in turn implies that $\Gamma < N_G(g)$.

If $\dim Z_G(\gamma) \geq 2$ for all $\gamma \in \Gamma$, then Γ is again central and contained in a connected Abelian subgroup. Thus the same argument as above applies. Thus we may assume that there exists $\gamma \in \Gamma$ with $\dim N_G(\gamma) = 1$. If $N_G(\gamma)$ is connected, then $\Gamma \subset N_G(\gamma)$ and we obtain a contradiction to lemma 5.2. If $N_G(\gamma)$ is not connected, then lemma 5.3 asserts that $Z_G(\gamma)^0 \subset G'$. But this is also not possible: use the 1-dimensional normalizer argument to see that there exists an element $\gamma' \in \Gamma \cap G'$. However, above we have already ruled out this possibility. \square

This finishes the proof of the main theorem 1.1 up to the proof 4.4 and the $\mathbb{C}^* \times \mathbb{C}^*$ -action argument.

6. PROOF OF THE $\mathbb{C}^* \times \mathbb{C}^*$ -ACTION ARGUMENT

Lemma 6.1. *Suppose that $T \cong (\mathbb{C}^*)^2$ acts on X . If E_i is an irreducible component of E , then T acts almost transitively on E_i .*

Proof. Suppose that T does not act almost transitively on E_i . Then proposition 2.5 asserts that the generic T -orbit $O \subset E_i$ is 1-dimensional. We know that $O \cong \mathbb{C}^*$ or that O is isomorphic to a 1-dimensional compact torus. In either case, $\dim \text{Aut}(O) = 1$, so that there is a 1-dimensional kernel $T' = \ker(T \rightarrow \text{Aut}(O))$. If $x \in O$ is any point, then linearize the T' -action at x (note that O does not meet the singular locus of E_i). The linearization shows that either T' acts trivially on E_i , or that the T' -orbits are transversal to O . The first case is ruled out by proposition 2.5. The second case is ruled out by the assumption that T does not act almost transitively. In each case we obtain a contradiction. \square

Lemma 6.2. *There exists an irreducible component E_i of E which is rational.*

Proof. First, we claim that the T -fixed point set is not empty. Identifying T with $(\mathbb{C}^*)^2$, we write $T = T_1.T_2$. By lemma 2.5 we know that $\text{Fix}(T_1)$ consists of 2 points, or it is isomorphic to \mathbb{P}^1 . In both cases we are finished if we note that T is Abelian, so that T stabilizes $\text{Fix}(T_1)$.

Second, choose a point x in the T -fixed point set and a number i such that $x \in E_i$. We will show that there is a T -stable rational curve $C \subset E_i$ with $x \in C$. Then, if \hat{E}_i is a minimal desingularization and \hat{C} a component of the preimage of C which is not mapped to a point, \hat{C} is still T -stable, and contains a T -fixed point. The classification of the smooth almost homogeneous surfaces yields that \hat{E}_i contains a T -fixed point only if it is rational. See e.g. [HO80, p. 92f].

In order to construct C , linearize the T -action at x . Denote the weights of T_1 by (a, b, c) and the weights of T_2 by (d, e, f) . Recall from proposition 2.5 that at most one weight in each triple is 0. If necessary, use an embedding $\mathbb{C}^* \rightarrow T$, $\lambda \mapsto (\lambda^{-d}, \lambda^a)$ to find an action of $T' \cong \mathbb{C}^*$ on X , fixing x and having weights $(0, g, h)$. Now let C be the T' -fixed point curve. Since $c_1(\mathcal{O}_X(E_i)) = 0$, C must be contained in E_i . \square

Now we finish the proof of the $\mathbb{C}^* \times \mathbb{C}^*$ -action argument 4.4.

Proof. Let $E_i \subset E$ be a rational component, and E_j a component different from E_i . If $\delta : \hat{E}_i \rightarrow E_i$ is a minimal resolution of the singularities, then, since $b_2(X) = 0$, $c_1(\delta^*(\mathcal{O}_X(E_j))) = 0$, but $\delta^*(E_j)$ is an effective divisor. This contradicts \hat{E}_i being rational. \square

7. CONSIDERATIONS CONCERNING THE \mathbb{C}^* -ACTION ARGUMENT

The remainder of the paper is devoted to proving proposition 4.6. As was indicated at the beginning of section 2, this completes the proof of our main theorem, i.e., that X is not almost homogeneous.

Accepting the situation presented to us by proposition 4.6, we operate here under the assumptions that Ω has the Betti type of S^1 and that there exists an effective \mathbb{C}^* -action on X . We begin by deriving some topological consequences of the assumption on the Betti type of Ω .

7.1. Topological constraints.

Proposition 7.1. *The divisor E is irreducible and not normal. Its normalization \tilde{E} has only rational singularities and the minimal desingularization \hat{E} is rational. In particular, \hat{E} is \mathbb{Q} -factorial.*

Proof. Since Ω has Betti-type S^1 , it follows from proposition 2.3 that $b_4(E) = 1$, i.e. E is irreducible. Suppose that E is normal and let $\pi : \hat{E} \rightarrow E$ be the minimal desingularization. Since $b_3(E) = 0$ we have $b_3(\hat{E}) = 0$ (see the proof of proposition 4.8), hence $b_1(\hat{E}) = 0$, and \hat{E} is Kähler.

Since $K_{\hat{E}} \subset \pi^*(K_E)$ and $\pi^*(K_E)$ is linearly equivalent to 0, we have $\kappa(\hat{E}) \leq 0$, and by $b_3(\hat{E}) = 0$, \hat{E} is either rational or birational to a K3-surface or an Enriques surface. But because \hat{E} possesses a \mathbb{C}^* -action, the latter cases are excluded. So \hat{E} (and hence E) are rational surfaces. As a consequence note that $R^1\pi_*(\mathbb{Q}) = 0$ by Leray's spectral sequence and

$$H^1(\hat{E}, \mathbb{Q}) = H^2(E, \mathbb{Q}) = 0.$$

Thus $H^2(\hat{E}, \mathbb{Q}) = H^0(E, R^2\pi_*(\mathbb{Q}))$ and we conclude that $H^2(\hat{E}, \mathbb{Q})$ is generated by the π -exceptional curves.

Now take an ample divisor A on \hat{E} . Then we find $m \in \mathbb{N}$, $\lambda_i \in \mathbb{Z}$ and π -exceptional curves $C_i \subset \hat{E}$ such that

$$c_1(\mathcal{O}_{\hat{E}}(mA)) = c_1\left(\mathcal{O}_{\hat{E}}\left(\sum \lambda_i C_i\right)\right).$$

Since \hat{E} is rational we have the linear equivalence $mA = \sum \lambda_i C_i$ which is absurd.

Consequence: E is not normal.

Let $\nu : \tilde{E} \rightarrow E$ be the normalization, and $\pi : \hat{E} \rightarrow \tilde{E}$ the minimal desingularization. Using the formula $\omega_{\tilde{E}} = \nu^*(\omega_E) - \tilde{N}$ and the fact that \tilde{N} is an effective Weil-divisor supported on the preimage of the non-normal locus (observe that E is Gorenstein!), the “old” arguments still apply and give the rationality of E .

In order to show that \tilde{E} has only rational singularities, we check

$$R^1\pi_*(\mathcal{O}_{\hat{E}}) = 0.$$

Since $H^1(\mathcal{O}_{\hat{E}}) = 0$, Leray's spectral sequence yields an embedding

$$H^0(R^1\pi_*(\mathcal{O}_{\hat{E}})) \rightarrow H^2(\mathcal{O}_{\hat{E}}).$$

Now \tilde{E} is Cohen-Macaulay and therefore $H^2(\mathcal{O}_{\tilde{E}}) \cong H^0(\omega_{\tilde{E}})$. Since $\omega_{\tilde{E}} \subset \nu^*(\omega_E)$ and $\omega_{\tilde{E}} \neq \nu^*(\omega_E) = \mathcal{O}_E$, we have $H^2(\mathcal{O}_{\tilde{E}}) = 0$. Therefore $R^1\pi_*(\mathcal{O}_{\hat{E}}) = 0$. \square

Notation 7.2. Let $\nu : \tilde{E} \rightarrow E$ be the normalization and $\pi : \hat{E} \rightarrow \tilde{E}$ be the minimal desingularization. Write $\delta := \pi \circ \nu$. Let $N \subset E$ be the non-normal locus and $\tilde{N} := \nu^{-1}(N)$.

Proposition 7.3. *The following Betti-numbers are equal: $b_0(N) = b_0(\tilde{N})$ and $b_1(N) = b_1(\tilde{N})$.*

Proof. We use the following Mayer-Vietoris sequence for reduced cohomology:

$$\dots \rightarrow \tilde{H}^q(E; \mathbb{Q}) \rightarrow \tilde{H}^q(\tilde{E}; \mathbb{Q}) \oplus \tilde{H}^q(N; \mathbb{Q}) \rightarrow \tilde{H}^q(\tilde{N}; \mathbb{Q}) \rightarrow \tilde{H}^{q+1}(E; \mathbb{Q}) \rightarrow \dots$$

(see [BK82, prop. 3.A.7 on p. 98] for information about this sequence).

So $h^0(\tilde{N}; \mathbb{Q}) = h^0(N; \mathbb{Q})$ since $H^1(E; \mathbb{Q}) = 0$ by proposition 2.3. Furthermore $b_1(\tilde{E}) = b_1(\tilde{N}) - b_1(N)$, since $H^2(E; \mathbb{Q}) = 0$. \square

Lemma 7.4. *The space N is connected.*

Proof. Using the fact that E is Cohen-Macaulay, by [Mor82, p. 166, 3.34(2)] there exists an exact sequence

$$0 \longrightarrow \mathcal{O}_E \longrightarrow \nu_*(\mathcal{O}_{\tilde{E}}) \longrightarrow \omega_E^{-1} \otimes \omega_N \longrightarrow 0$$

Since $c_1(\nu^*(\omega_E)) = 0$ so that $\nu^*(\omega_E) = \mathcal{O}_{\tilde{E}}$,

$$0 \longrightarrow \omega_E \longrightarrow \underbrace{\omega_E \otimes \nu_*(\mathcal{O}_{\tilde{E}})}_{=\nu_*\nu^*(\omega_E)=\nu_*(\mathcal{O}_{\tilde{E}})} \longrightarrow \omega_N \longrightarrow 0.$$

is also exact. Again using [Mor82, p. 166, 3.34(2)] we see that N is Cohen-Macaulay so that Serre duality holds. Thus, the associated long cohomology sequence gives

$$\dots \longrightarrow \underbrace{H^1(\tilde{E}, \mathcal{O}_{\tilde{E}})}_{=0} \longrightarrow \underbrace{H^1(N, \omega_N)}_{\cong H^0(N, \mathcal{O}_N)} \longrightarrow \underbrace{H^2(E, \omega_E)}_{\cong H^0(E, \mathcal{O}_E) \cong \mathbb{C}} \longrightarrow \dots$$

so that $h^0(N, \mathcal{O}_N) \leq 1$, and the reduced subspace N_{red} is connected. \square

7.2. Orbits of the \mathbb{C}^* -action. For the sake of completeness we outline some known facts on \mathbb{C}^* -actions on rational surfaces: the orbits are always constructible. As a consequence we see that E necessarily contains attractive and repulsive fixed points, a fact that will be crucial in the sequel. More information can be found in the works of Białynicki-Birula and of Sommese (see e.g. [BBS85]).

Lemma 7.5. *If H is a linear algebraic group and $\iota : \mathbb{C}^* \rightarrow H$ is a holomorphic map which is a (set-theoretical) group morphism, then the image $\iota(\mathbb{C}^*)$ is a closed algebraic subgroup of H .*

Proof. Let $J < H$ be the Zariski closure of $\iota(\mathbb{C}^*)$, in H . It is immediate that J is Abelian and, since it is affine, $J \cong \mathbb{C}^n \times (\mathbb{C}^*)^m$.

Now use the fact that there is no non-constant holomorphic group morphism from \mathbb{C}^* to \mathbb{C} , and that any group morphism from \mathbb{C}^* to \mathbb{C}^* is given by $z \mapsto z^k$. \square

Lemma 7.6. *Let S be a (possibly singular) irreducible surface with a holomorphic action of \mathbb{C}^* . Suppose that the minimal desingularization \tilde{S} is rational. Then:*

1. *All \mathbb{C}^* -orbits are constructible, and their closures are rational. In particular, for all $x \in S$, the limits $\lim_{\lambda \in \mathbb{C}^*, \lambda \rightarrow 0} \lambda.x$ and $\lim_{\lambda \in \mathbb{C}^*, \lambda \rightarrow \infty} \lambda.x$ exist.*
2. *If $F \subset S$ is the set of the \mathbb{C}^* -fixed points. Then there are two components F_0 and F_∞ of F and a Zariski-open set $U \subset S$ such that for all $x \in U$:*

$$\lim_{\lambda \in \mathbb{C}^*, \lambda \rightarrow 0} \lambda.x \in F_0, \quad \lim_{\lambda \in \mathbb{C}^*, \lambda \rightarrow \infty} \lambda.x \in F_\infty.$$

Proof. Suppose for the moment that S was smooth. Then $\text{Aut}(S)$ is linear algebraic and acts algebraically; this is a consequence of the fact that, since $b_1(S) = 0$, S can be equivariantly embedded into some \mathbb{P}_n —see [Bla56] for a proof. By lemma 7.5, any closed subgroup $\mathbb{C}^* < \text{Aut}(S)$ is linear algebraic and acts algebraically. In particular, \mathbb{C}^* -orbits are constructible. It follows from Borel's fixed point theorem (see e.g. [HO80, p. 32]) that all \mathbb{C}^* -stable curves in S contain fixed points. This already proves (1).

In order to prove assertion (2), embed $\mathbb{C}^* \rightarrow \mathbb{P}_1$ in the usual way. Then there exists a rational morphism

$$\phi : \mathbb{P}_1 \times S \dashrightarrow S.$$

Since the set of fundamental points of ϕ^{-1} is of codimension ≥ 2 , there exists an open set $U \subset S$ such that $\phi|_{\mathbb{P}_1 \times U}$ is regular. Consequence: for all $x \in U$ the limits $\lim_{\lambda \in \mathbb{C}^*, \lambda \rightarrow 0} \lambda.x$ and $\lim_{\lambda \in \mathbb{C}^*, \lambda \rightarrow \infty} \lambda.x$ exist. Set $F_0 = \overline{\phi(0 \times U)}$ and $F_\infty := \overline{\phi(\infty \times U)}$.

If S is singular, then let $\delta : \hat{S} \rightarrow S$ be an equivariant resolution and find \hat{F}_0 , \hat{F}_∞ and $\hat{U} \subset \hat{S}$ as above. It is sufficient to set $F_0 := \delta(\hat{F}_0)$, $F_\infty := \delta(\hat{F}_\infty)$ and $U := \delta(\hat{U}) \cap S_{reg}$, where S_{reg} is the (open) set of regular points in S . \square

8. THE CASE OF A DISCRETE FIXED POINT SET

Our goal here is to prove the \mathbb{C}^* -action argument under the assumption that the set F of \mathbb{C}^* -fixed points is discrete, i.e. $F = \{F_0, F_\infty\}$. It is shown that the discreteness assumption, which is made throughout this section, is contrary to $b_1(N) = b_1(\tilde{N})$ (see proposition 8.21).

Notation 8.1. In the sequel U_0 and U_∞ are disjoint linearizing neighborhoods of the points F_0 and F_∞ in X .

8.1. Symmetry lemmas. Here we prove several lemmas which show that the situations at F_0 and at F_∞ are very similar. First, we investigate:

8.1.1. Weights of the \mathbb{C}^* -actions on $T_{F_0}X$ and $T_{F_\infty}X$. The main result of this section is

Proposition 8.2. *The locally closed spaces $E \cap U_0$ and $E \cap U_\infty$ are reducible. One of the components of each space is smooth, and the \mathbb{C}^* -action has a totally attractive resp. repulsive fixed point there.*

Since the proof is somewhat lengthy, and we will use a number of partial results later, we subdivide the proof into a sequence of lemmas and corollaries.

Lemma 8.3. *After swapping F_0 and F_∞ and, if necessary, replacing the \mathbb{C}^* -action by $(\mathbb{C}^*)^{-1}$, the weights of the \mathbb{C}^* -action on the tangent space $T_{F_0}X$ have the following signs: $(+ + -)$.*

Proof. By symmetry, we only have to exclude the following distribution of signs of the weights of the \mathbb{C}^* -action on $T_{F_0}X$ and $T_{F_\infty}X$:

$(+ + +), (+ + +)$: This contradicts lemma 7.6.

$(+ + +), (- - -)$: Let $x \in E$ be a generic point. Choose a sequence $(x_n)_{n \in \mathbb{N}} \subset \Omega$ such that $\lim_{n \rightarrow \infty} x_n = x$. Then there exist numbers $\lambda, \lambda' \in \mathbb{C}^*$ such that $\lambda.x \in U_0$ and $\lambda'.x \in U_\infty$ and there exists an $n \in \mathbb{N}$ such that $\lambda.x_n \in U_0$, and $\lambda'.x_n \in U_\infty$. Linearization shows that $\lim_{\lambda \rightarrow 0} \lambda.x_n = F_0$ and $\lim_{\lambda \rightarrow \infty} \lambda.x_n = F_\infty$ so that $C := \overline{\mathbb{C}^*.x_n}$ is a closed curve in X and $C \cap E = F_0 \cup F_\infty$. But then $E.C \geq 2$, a contradiction to $H^2(X, \mathbb{Z}) = 0$. \square

Assume from now on that the \mathbb{C}^* -action on $T_{F_0}X$ has weights of type $(+ + -)$. Let x, y and z be coordinates for the associated weight spaces. By the linearization theorem 2.4, we can view x, y and z as giving local coordinates on U_0 . After performing a \mathbb{C}^* -equivariant change of coordinates on $T_{F_0}X$, we may assume that the unit ball in $T_{F_0}X$ is contained in the image of U_0 .

Corollary 8.4. *After swapping F_0 and F_∞ and, if necessary, replacing the \mathbb{C}^* -action by $(\mathbb{C}^*)^{-1}$, the weights of the \mathbb{C}^* -action on the tangent space $T_{F_0}X$ have signs $(++-)$ and the locally closed subspace $E \cap U_0$ is reducible. One of the components of $E \cap U_0$ is smooth, and the \mathbb{C}^* -action has a totally attracting fixed point there.*

Proof. Since $E \cap U_0$ is closed in U_0 and contains infinitely many \mathbb{C}^* -stable curves containing F_0 , we have that $\{z = 0\} \cap U_0 \subset E$.

Since $\{z = 0\}$ is smooth, in order to see that $E \cap U_0$ is reducible, it suffices to show $E \cap U_0$ is not normal. Since E is a hypersurface in X , it is Cohen-Macaulay, and it follows from Serre's criterion that the non-normal locus $N \subset E$ must be of codimension 1. By lemma 7.6, N contains \mathbb{C}^* -fixed points. If N contains F_0 , then we can stop here.

Otherwise, if N contains F_∞ , but not F_0 , then the signs of the weights at F_∞ are necessarily $(--+)$; we show this by ruling out all other possibilities:

- $(+++)$: does not occur, or else obtain a contradiction to lemma 7.6.
- $(++-)$: similarly
- $(---)$: then $\lim_{\lambda \rightarrow 0} \lambda x$ would not exist for generic $x \in N$. Since every 1-dimensional component of N contains a \mathbb{C}^* -fixed point, it is rational. Thus, the limit exists on the normalization of N which in turn implies that the limit exists on N .

In this case swap F_0 and F_∞ and start anew. \square

For the remainder of this section, fix F_0 and F_∞ so that we are in the situation of the above corollary.

Notation 8.5. Let $E_{0,i}$ denote the irreducible components of $E \cap U_0$ and let $E_{0,0}$ be the smooth component in the preceding corollary.

Lemma 8.6. *Choose numbers a, b and $c \in \mathbb{N}^+$ and let the group \mathbb{C}^* act on \mathbb{C}^3 by $\lambda : (x, y, z) \rightarrow (\lambda^a x, \lambda^b y, \lambda^{-c} z)$. Let $S \subset \mathbb{C}^3$ be an irreducible \mathbb{C}^* -stable divisor with $S \neq \{z = 0\}$, but $S \cap \{z = 0\} \neq \emptyset$. Then $\{x = y = 0\} \subset S$.*

Proof. Choose a point $s \in \{z = 0\} \cap S$, s. Let $s_n = (x_n, y_n, z_n)$ be a sequence with $z_n \neq 0$ and $\lim s_n = s$. Choose $\lambda_n \in (z_n)^{\frac{1}{c}}$. Note that $\lim \lambda_n = 0$. Then $\lambda_n \cdot s_n \in E$, and $\lim \lambda_n s_n = \lim(\lambda_n^a x_n, \lambda_n^b y_n, 1) = (0, 0, 1)$. \square

Corollary 8.7. *Every component $E_{0,i}$ ($i \neq 0$) contains $\{x = y = 0\}$.*

Lemma 8.8. *If the signs of the weights of the \mathbb{C}^* -action on $T_{F_\infty}X$ are all negative, then there exists a curve $C^- \subset E$ satisfying*

1. $C^- \cap E_{0,0}$ is a curve (i.e. $\dim C^- \cap E_{0,0} = 1$) and
2. $F_\infty \not\subset C^-$

Proof. By corollary 8.4, $E \cap U_0$ is reducible, and by corollary 8.7, $\{x = y = 0\} \subset E \cap U_0$. The z -axis is the weight-space to the negative weight, so that $\lim_{\lambda \rightarrow \infty} (0, 0, 1) = F_0$. But the limit $\lim_{\lambda \rightarrow 0} (0, 0, 1)$ exists; this is because \hat{E} is rational, and the limit exists there. Due to the negative weights it is impossible that $\lim_{\lambda \rightarrow 0} (0, 0, 1) = F_\infty$. Thus $\lim_{\lambda \rightarrow 0} (0, 0, 1) = F_0$ and for all λ sufficiently small $\lambda \cdot (0, 0, 1) \in E_{0,0}$. \square

Lemma 8.9. *The weights of the \mathbb{C}^* -action on $T_{F_\infty}X$ have signs $(--+)$.*

Proof. It is clear that at least two of the signs must be negative —this is because for generic $x \in E$, $\lim_{\lambda \rightarrow \infty} \lambda x = F_0$. Now suppose the weights were (a, b, c) which were all negative. The weights of the \mathbb{C}^* -action on $T_{F_0}E_{0,0}$ shall be denoted by d and e . We consider the weighted projective spaces $\mathbb{P}_{(-a,-b,-c)}(T_{F_\infty}X)$ and $\mathbb{P}_{(d,e)}(T_{F_0}E_{0,0})$. These are parameter spaces for \mathbb{C}^* -stable curves in X and E passing through F_0 or F_∞ , respectively.

The analytic subspace $E \cap U_\infty$ gives a *closed* subspace in the weighted projective space $\mathbb{E} \subset \mathbb{P}_{(-a,-b,-c)}(T_{F_\infty}X)$ parameterizing curves in $E \cap U_\infty$. We will now construct a map from \mathbb{E} to $\mathbb{P}_{(d,e)}(T_{F_0}E_{0,0})$.

First fix some notation: let $\Lambda_0 : E_{0,0} \rightarrow T_{F_0}E_{0,0}$ and $\Lambda_\infty : U_\infty \rightarrow T_{F_\infty}X$ be the linearizing maps, and let $\pi_0 : T_{F_0}E_{0,0} \setminus \{0\} \rightarrow \mathbb{P}_{(d,e)}(T_{F_0}E_{0,0})$ be the canonical projection.

Given an arbitrary point $x \in \mathbb{E}$, there exists a neighborhood $U(x) \subset \mathbb{E}$ and a section $\sigma : U(x) \rightarrow \Lambda_\infty(U_\infty) \subset T_{F_\infty}X$. After shrinking $U(x)$, if necessary, there is a $\lambda \in \mathbb{C}^*$ such that $(\lambda \circ \sigma)(U(x)) \subset E_{0,0} \setminus \{F_0\}$: simply choose a λ such that $(\lambda \circ \sigma)(v) \in E_\infty$ and set $U'(x) := \lambda^{-1}((\lambda \circ \sigma)(U(x)) \cap E_{0,0})$; this is the shrinkage that might be unavoidable. This way we obtain a map $\iota_x := (\pi_0 \circ \lambda \circ \sigma) : U(x) \rightarrow \mathbb{P}_{(d,e)}(T_{F_0}E)$.

In order to obtain a global map $\iota : \mathbb{E} \rightarrow \mathbb{P}_{(d,e)}(T_{F_0}E)$, it suffices to show that ι_x does not depend on the choice of σ and λ . Indeed, choosing different σ' and λ' then for all $y \in U(x)$ there is a unique number $\lambda_y \in \mathbb{C}^*$ such that $(\lambda \circ \sigma)(y) = \lambda_y(\lambda' \circ \sigma')(y)$. This already shows that $(\pi_0 \circ \lambda \circ \sigma)(y) = (\pi_0 \circ \lambda' \circ \sigma')(y)$, and the existence of the global map ι is shown. It is obvious that ι is injective.

This is how we make use of ι : by lemma 8.8, $\pi(C^- \cap E_{0,0}) \setminus \{F_0\}$ is not contained in the image of ι , so that the image must be contained in $\mathbb{P}_{(d,e)}(T_{F_0}E) \setminus (\pi(C^- \cap E_{0,0} \setminus \{F_0\})) \cong \mathbb{C}$. By the maximum principle, the image must be a point. A contradiction to ι being injective. \square

Corollary 8.10. *$E \cap U_\infty$ is reducible. There exists a smooth component $E_{\infty,0}$ with totally repulsive fixed point.*

Proof. The existence of $E_{\infty,0}$ follows exactly as in corollary 8.4. Similarly, it follows from the argumentation of corollary 8.4 that $E \cap U_\infty$ is reducible, if we show that the non-normal locus $N \subset E$ intersects $E_{\infty,0}$.

Suppose this was not the case. Then, if $x \in E_{\infty,0}$, $\lim_{\lambda \rightarrow 0} \lambda x = F_0$, and the argumentation used in the proof of lemma 8.9 yields a contradiction. \square

Notation 8.11. In analogy to the notation introduced above, let $E_{\infty,i}$ denote the irreducible components of $E \cap U_\infty$ and let $E_{\infty,0}$ be the smooth component whose existence is asserted by the preceding corollary.

8.1.2. Loops. Now turn to the non-normal locus of $N \subset E$. The following is an important notion:

Notation 8.12. Call a \mathbb{C}^* -stable curve $C \subset E$ a “loop”, if for a generic point $x \in C$:

$$\lim_{\lambda \rightarrow 0} \lambda x = \lim_{\lambda \rightarrow \infty} \lambda x$$

Lemma 8.13. *There is at most one loop containing F_0 , and at most one containing F_∞ . The number of loops containing F_0 and the number of loop containing F_∞ are equal.*

Proof. There is only one curve in $E \cap U_0$ containing a point x such that $\lim_{\lambda \rightarrow \infty} \lambda.x = F_0$ (namely the z -axis). The situation at F_∞ is similar.

Argue as in the proof of lemma 8.9, using a map

$$\mathbb{P}_{(a,b)}(T_{F_0}E_{0,0}) \rightarrow \mathbb{P}_{(c,d)}(T_{F_\infty}E_{\infty,0})$$

to exclude the possibility that there is no loop at F_0 and one at F_∞ or vice versa. \square

Lemma 8.14. *The number of irreducible components of $N \cap E_{0,0}$ equals the number of irreducible components of $N \cap E_{\infty,0}$.*

Proof. Decompose $N = N_L \cup \bigcup_i N_i$, where the N_L are loops and the N_i are other components. By lemma 8.13, the claim is true if one considers loops only. If N_i is one of the other components and $x \in N_i$ a generic point, then $\lim_{\lambda \rightarrow 0} \lambda.x = F_0$ and $\lim_{\lambda \rightarrow \infty} \lambda.x = F_\infty$ so that $N_i \cap E_{0,0}$ and $N_i \cap E_{\infty,0}$ are both irreducible components of $N \cap E_{0,0}$ and $N \cap E_{\infty,0}$, respectively. \square

8.2. Preparations: \mathbb{C}^* -action on normal surfaces. Let S be a smooth connected algebraic surface equipped with an algebraic \mathbb{C}^* -action with an attractive (resp. repulsive) fixed point F_∞ (resp. F_0). Let $U \subset S$ be as in lemma 7.6.

Notation 8.15. A curve $C = C_+ \cup C_1 \cup \dots \cup C_k \cup C_-$, $k \geq 0$, as in the following figure

$$F_0 \bullet \xrightarrow{C_+} \bullet_{p_1} \xrightarrow{C_1} \bullet_{p_2} \xrightarrow{C_2} \dots \xrightarrow{C_-} \bullet_{F_\infty}$$

which is invariant under the \mathbb{C}^* -action will be called an “external chain”.

A \mathbb{C}^* -fixed point different from F_0 and F_∞ is an “external fixed point”.

Lemma 8.16. *The complement of U in S is a union of external chains without common components.*

Proof. At every external fixed point the weights of the linearization must be of type $(+-)$. \square

Now allow S to have normal singularities at external fixed points. Applying the above argument to the desingularization \hat{S} and blowing back down, we have the same result.

Lemma 8.17. *If S is a normal compact rational surface equipped with an \mathbb{C}^* -action having a smooth point F_0 as a source and a smooth point F_∞ as a sink, then the complement $S \setminus U$ is a union of external chains.*

Corollary 8.18. *In the setting of the preceding lemma there are no loops in S .*

8.3. Computation of b_1 of curves.

Lemma 8.19. *Let C be the union of c rational curves having s singular points. Let $\nu : \tilde{C} \rightarrow C$ be the normalization of C and \tilde{s} be the number of points in $\nu^{-1}(C_{Sing})$. For convenience, let $\delta = \tilde{s} - s$. Then*

$$b_1(C) = 1 - c + \delta$$

Proof. Apply the Mayer-Vietoris sequence (see p. 11), where $E := C$, $N := C_{Sing}$, etc.:

$$0 = \underbrace{\tilde{h}^0(E)}_0 - \underbrace{(\tilde{h}^0(\tilde{E}) + \tilde{h}^0(N))}_{c-1} + \underbrace{\tilde{h}^0(\tilde{N})}_{\tilde{s}-1} - \underbrace{h^1(E)}_{b_1} + \underbrace{(h^1(\tilde{E}) + h^1(N))}_0.$$

□

8.4. The proof in the case where F is discrete.

Notation 8.20. An “outer orbit” is an orbit which flows in the opposite direction from the generic orbit, i.e. with source F_∞ and sink F_0 .

An irreducible \mathbb{C}^* -invariant curve C containing F_0 as a source and F_∞ as a sink is called a “crossing curve” if E is not locally irreducible at the generic point of C .

Proposition 8.21. *The case where F is discrete does not occur.*

Proof. Assume to the contrary and use the notation introduced above. Let n be the number of components in $N \cap E_{0,0}$. Then, by lemma 8.19,

$$b_1(N) = \begin{cases} n-1 & \text{if } N \text{ consists of crossing curves only} \\ n & \text{if } N \text{ contains an outer orbit} \\ n+1 & \text{if } N \text{ contains loops} \end{cases}$$

on the other hand, since the number of external chains in \tilde{E} plus the number of curves in $\tilde{U} \cap \tilde{N}$ is also n (note that there are n components of \tilde{N} containing \tilde{F}_0), we have

$$b_1(\tilde{N}) = n - 1.$$

It remains to show that the case where N only consists of crossing curves does not occur. In that case, since all components $E_{0,i}$, $i \neq 0$, contain $\{x = y = 0\}$ and N does not contain an outer orbit, there is only one such component $E_{0,1}$. Furthermore, $E_{0,0} \cap E_{0,1}$ consists of closures of \mathbb{C}^* -orbits which flow from F_0 to F_∞ .

Observe that in this situation there can not be external chains in \tilde{E} : If $p \in \tilde{N}$ is an external fixed point, then it is the intersection point of a curve flowing *into* p with a curve flowing *out of* p . But if $\nu(p) = F_0$, then there all curves in $N = \nu(\tilde{N})$ flow *out of* $\nu(p)$, and we would have contradiction.

The analogue holds if $\nu(p) = F_\infty$. □

9. PROOF OF THE \mathbb{C}^* -ACTION ARGUMENT IF F IS A CURVE

If F is a curve, then $F \cong \mathbb{P}_1$ by proposition 2.5. It is very easy to calculate $b_1(N)$. However, we first have to show that $F \subset N$.

Lemma 9.1. *Suppose $\dim F = 1$. If $x \in F$ is an arbitrary point, and U a linearizing neighborhood of \mathbb{C}^* about x , then the weights of the \mathbb{C}^* -action on $T_x X$ have signs $(0 + -)$, and if x, y and z are coordinates associated to the weight spaces, then $\{yz = 0\} \subset E \cap U$.*

Recall from the theorem on linearization 2.4 that x, y and z can be viewed to give coordinates on U .

Proof. We have to exclude the possibility that the signs of the non-zero weights are equal. Suppose they were both positive. Then for a point $y \in (E \setminus F) \cap U$, the limit $\lim_{\lambda \rightarrow \infty} \lambda y$ would not exist in F . This contradicts the fact that F is the full \mathbb{C}^* -fixed point set.

Now it is a direct consequence of lemma 7.6 that $\{zy = 0\} \subset E \cap U$. Note that $F_0 = F_\infty = F$. \square

Lemma 9.2. *If $F \cong \mathbb{P}_1$, then $b_1(N) = (\# \text{ of irreducible components of } N) - 1$.*

Proof. Decompose $N = F \cup \bigcup_i N_i$. We show by induction that $b_1(F \cup \bigcup_{i=1 \dots k} N_i) = k$.

Start, $k = 0$: It is clear that $b_1(F) = b_1(\mathbb{P}_1) = 0$.

Step: Since for all $x \in N_i$, the limits $\lim_{\lambda \rightarrow 0} \lambda x$ and $\lim_{\lambda \rightarrow \infty} \lambda x$ exist and are \mathbb{C}^* -fixed, i.e. contained in F , there are only two possibilities:

1. $\lim_{\lambda \rightarrow 0} \lambda x = \lim_{\lambda \rightarrow \infty} \lambda x$ for all $x \in N_k$, i.e. N_k is a loop (see the notation 8.12). Then $b_1(N_k) = 1$ and $N_k \cap F$ is a single point.
2. $\lim_{\lambda \rightarrow 0} \lambda x \neq \lim_{\lambda \rightarrow \infty} \lambda x$. Then the normalization $\tilde{N}_k \rightarrow N_k$ is injective and thus $b_1(N_i) = 0$. Furthermore $N_k \cap F$ are two points.

In any case the Mayer-Vietoris sequence associated to the decomposition $F \cup \bigcup_{i=1 \dots k} N_i = (F \cup \bigcup_{i=1 \dots k-1} N_i) \cup N_k$ shows directly that

$$b_1(F \cup \bigcup_{i=1 \dots k} N_i) = b_1(F \cup \bigcup_{i=1 \dots k-1} N_i) + 1 = k.$$

\square

Again we use a description of the rational surface \tilde{E} to calculate $b_1(\tilde{N})$ and finally to derive a contradiction.

Lemma 9.3. *There exists a surjective morphism with connected fibers $\phi : \tilde{E} \rightarrow \mathbb{P}_1$ such that the induced \mathbb{C}^* -action on \mathbb{P}_1 is trivial.*

Proof. Let C and C' be two generic irreducible \mathbb{C}^* -stable curves in \tilde{E} . Since the \mathbb{C}^* -fixed point set in \tilde{E} does not contain a totally attractive fixed point (this is because there is non in E), C and C' are disjoint and do not intersect the singular locus of \tilde{E} .

Since \tilde{E} is rational, C and C' are linearly equivalent as divisors and yield the desired map. \square

Lemma 9.4. *The situation that $F \cong \mathbb{P}_1$ does not occur.*

This finishes the proof of the \mathbb{C}^* -action argument 4.6.

Proof. We show that $b_1(\tilde{N}) = (\# \text{ of irreducible components of } N) - 2$, contradicting lemma 9.2. In accordance with the notation and the results of lemma 7.6, let F_0 and $F_\infty \subset \tilde{E}$ denote the \mathbb{C}^* -fixed point curves. These are sections for $\phi : \tilde{E} \rightarrow \mathbb{P}_1$, and contained in the preimage $\nu^{-1}(F)$. In particular, $F_0 \cup F_\infty \subset \tilde{N}$.

Again decompose $N = F \cup \bigcup_i N_i$. Set $\tilde{N}_i := \nu^{-1}(N_i)$.

First, we show that $\tilde{N}_i \cap F_0$ and $\tilde{N}_i \cap F_\infty$ are both single points. Realize from the description of lemma 9.1 that $\nu|_{F_0} : F_0 \rightarrow N$ is injective, and note that if $y \in N_i$ is a generic point, then

$$\tilde{N}_i \cap F_0 \subset \nu^{-1}(\lim_{\lambda \rightarrow 0} \lambda y) \cap F_0.$$

The expression on the right hand side denotes a single point. A similar argumentation holds for F_∞ .

Second, we claim that $b_1(\tilde{N}_i) = 0$. Recall that \tilde{N}_i are \mathbb{C}^* -stable, and do not contain F_0 or F_∞ as an irreducible component. Thus, the irreducible components of \tilde{N}_i are irreducible components of ϕ -fibers.

This is sufficient information to apply the Mayer-Vietoris sequence. For brevity, let $k := (\# \text{ of irreducible components of } N)$. The beginning of the cohomological Mayer-Vietoris sequence associated to the decomposition $\tilde{N} = (F_0 \cup F_\infty) \cup (\bigcup_i \tilde{N}_i)$ yields:

$$\underbrace{h^0(\tilde{N})}_{=1} - \underbrace{(h^0(F_0 \cup F_\infty))}_{=2} + \underbrace{h^0(\bigcup_i \tilde{N}_i)}_{\geq h^0(\bigcup_i N_i) = k-1} + \underbrace{h^0((F_0 \cup F_\infty) \cap (\bigcup_i \tilde{N}_i))}_{=2(k-1)} - h^1(\tilde{N}) = 0.$$

In other words, $h^1(\tilde{N}) \leq k - 2$. \square

10. SOME FURTHER REMARKS

Proposition 10.1. *There is no non-zero three-form on X , i.e. $H^0(K_X) = 0$.*

Proof. Let $\sigma \in H^0(\Omega_X^3)$. Then $d\sigma = 0$. Since $H^3(X; \mathbb{Q}) = 0$, de Rham's theorem exhibits a \mathcal{C}^∞ -form η of degree 2, such that $\sigma = d\eta$. Thus by Stokes

$$\int_X \sigma \wedge \bar{\sigma} = \int_X d\eta \wedge \bar{\sigma} = \int_X d(\eta \wedge \bar{\sigma}) - \int_X \eta \wedge d\bar{\sigma} = 0.$$

Hence $\sigma = 0$. \square

Corollary 10.2.

1. $H^3(X, \mathcal{O}_X) = 0$;
2. $H^0(K_X \otimes L) = 0$ for generic $L \in \text{Pic}(X)$;
3. $h^1(X, \mathcal{O}_X) = h^2(X, \mathcal{O}_X) + 1$;
4. $\text{Pic}(X) = \text{Pic}^0(X) \simeq H^1(X, \mathcal{O}_X)$ is a positive dimensional complex vector space.

Observe that it is not at all clear that $\kappa(X) = -\infty$. The proof of this fact seems to be as complicated as to show that there are no divisors on X at all (notice that K_X cannot be torsion by the above results!). See proposition 10.4 below.

We now prove the analogous statement to our main theorem for the cotangent bundle.

Proposition 10.3. *We have $h^0(\Omega_X^1) \leq 2$.*

Proof. Since X has no meromorphic functions, we have $h^0(\Omega_X^1) \leq 3$, moreover if $h^0(\Omega_X^1) = 3$, then any basis of 1-forms ω_i is linearly independent at the general point:

$$\omega_1 \wedge \omega_2 \wedge \omega_3 \neq 0$$

as a section of $K_X = \Omega_X^3$. This contradicts proposition 10.1. \square

Again we can conclude that $h^0(\Omega_X^1 \otimes L) \leq 2$ for the general line bundle L . Finally we can prove statement (B) of the introduction under some further assumptions:

Proposition 10.4.

1. Assume that X has no compact curves. Then $H^0(\Omega_X^1 \otimes L) = 0$ for all line bundles L , and $h^0(TX \otimes L) \leq 1$.
2. Assume that X has no divisors. Then $h^0(\Omega_X^1 \otimes L) \leq 2$ for all line bundles L .

Proof. (1) Let $\omega \in H^0(\Omega_X^1 \otimes L)$. Suppose $\omega \neq 0$. Then there is a divisor D (possibly empty) such that the section $\omega \in H^0(\Omega_X^1 \otimes L \otimes \mathcal{O}_X(-D))$ has no zeroes in codimension 1, hence has only finitely many zeroes by our assumption. Thus

$$c_3(\Omega_X^1 \otimes L \otimes \mathcal{O}_X(D)) = c_3(\Omega_X^1) \geq 0.$$

But $c_3(X) = -c_3(\Omega_X^1) = 2$, contradiction. The inequality $h^0(TX \otimes L) \leq 1$ is an immediate consequence from $H^0(\Omega_X^1 \otimes L) = 0$.

(2) Assume $h^0(\Omega_X^1 \otimes L) = 3$ and take a basis (ω_i) . Then $\omega_1 \wedge \omega_2 \wedge \omega_3$ is a non-zero section of $K_X \otimes (3L)$. Since there are no divisors on X , the ω_i are linearly independent everywhere, hence $\Omega_X^1 \otimes L \simeq \mathcal{O}_X^{\oplus 3}$. This contradicts again $c_3(X) = 2$. \square

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ALAN T. HUCKLEBERRY, RUHR-UNIVERSITÄT BOCHUM, FAKULTÄT FÜR MATHEMATIK, 44780 BOCHUM, GERMANY, ahuck@cplx.ruhr-uni-bochum.de

STEFAN KEBEKUS, LEHRSTUHL MATHEMATIK VIII, UNIVERSITÄT BAYREUTH, 95440 BAYREUTH, GERMANY, stefan.kebekus@uni-bayreuth.de

THOMAS PETERNELL, LEHRSTUHL MATHEMATIK I, UNIVERSITÄT BAYREUTH, 95440 BAYREUTH,
GERMANY, peternel@btm8x1.mat.uni-bayreuth.de